Changes in the age-at-death distribution in low mortality countries: A nonparametric approach

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Paper submitted for the XXVI IUSSP International Population Conference

Introduction

Over the course of the last century, we have witnessed major changes in the level of mortality in regions all across the globe. This remarkable mortality decrease has also been characterized by important changes in the distribution of ages at death, which inevitably led to substantial modifications in the shape of the survival curve over time. Measuring transformations in the survival curve or in the distribution of ages at death quickly became a subject of great interest among researchers, as their implications on societies are profound. Recently, more than 20 indicators were identified by Cheung et al. (2005), each aiming at quantifying either the central tendency or the dispersion (variability) of mortality across individuals. Given that all these indicators are tied to a parametrical statistical setting, which can imply fairly rigid modeling structure, a flexible nonparametric approach tailored to improve our monitoring of changes in the age at death distribution is worth considering.

Background

How long do we live in average? is probably one of the most recurrent demographic question. The traditional way of answering it is in terms of life expectancy at birth (average age at death in a life table). However, the late modal age at death (referred to hereafter as modal age at death) is another strong candidate who has received much recognition lately (Canudas-Romo, 2008; Cheung and Robine, 2007; Cheung et al., 2008, 2005; Kannisto, 2000, 2001, 2007; Paccaud et al., 1998; Robine, 2001). Lexis pioneer work Lexis (1877, 1878) on the concept of normal life durations considered the modal age at death to be the most central and natural characteristic of human longevity. Indeed, unlike the life expectancy at birth, the modal age is solely influenced by adult mortality and therefore much more sensitive to changes among the elderly population. In low mortality countries, it therefore consists in a useful tool to monitor and explain changes in the age-at-death distribution since most deaths are occurring at older ages.

Life table deaths have often been used to find accurate modal age at death estimates. Since the age distribution of life table deaths often tends to be irregular in the modal age area, Pearson (1902) recommended to interpolate a curve through the top of ordinates and to then use the maximum of this curve as the modal age. Canudas-Romo (2008), Cheung et al. (2008), Kannisto (2001), and Kannisto (2007) followed this advice and fit a quadratic polynomial to the life table deaths. Inspired by Lexis (Lexis, 1877, 1878) concept of normal life durations, Cheung and Robine (2007) fit a scaled normal model to the life table deaths (from five years before the observed modal age to the end of the life table). Both approaches involve parametric modelling which imposes a strict behavior stucture on deaths and therefore influence results.

How variable is the age we die at? is another relevant demographic question which has gained in popularity. Wilmoth and Horiuchi (1999) and Kannisto (2000) shed some interesting light on it by comparing various dispersion indicators in several countries. Kannisto also suggested a new mode-based indicator to monitor the dispersion of deaths above the modal age ($SD(M+)$). Since all these dispersion indicators are either computed with deaths extracted from life tables closed with a parametric model, or depend on the modal age, results are also affected by strict behavior stucture imposed on deaths.

Objectives

Concerns about the impact of parametric modelling on indicators have already been raised (Cheung and Robine, 2007; Cheung et al., 2008, 2005; Paccaud et al., 1998), but to our knowledge, the nonparametric path has never been explored extensively. The first objective of this paper is then to present a flexible nonparametric approach based on regression splines, specifically B-splines with penalties known as P-splines (Eilers and Marx, 1996), that has the potential to improve our monitoring of changes in the age-at-death distribution. We draw from previous work by Camarda (2009) and Currie et al. (2004) who specifically applied the method of P-splines to mortality data.

Secondly, we apply the suggested nonparametric method to a sample of four low mortality countries. Japan and France were selected because recent studies have identified them as countries where the modal age is the highest (Canudas-Romo, 2008; Cheung et al., 2008) and compression above the modal age has either stopped or slowed down substantially (Cheung et al., 2008; Robine et al., 2008). In these terms, the case of the United States shows great contrast with Japan and France (Canudas-Romo, 2008) and was therefore selected to explore differences accross low mortality populations further. Finally, Canada was chosen because very few studies on this topic have included it (Martel and Bourbeau, 2003; Nagnur, 1986) and comparisons with the United States should be informative.

Data and methods

Observed number of deaths Y_i and exposure to risk E_i by age $i = 10, \ldots, n$, and sex for Canada, France, Japan, and the United States are taken from the Human Mortality Database (HMD 2009). Complete life tables are also extracted from the HMD for comparison purposes. Data selected for France (1920-2005) pertains to the civilian population and starts after the influenza pandemic of 1918-19 to prevent sudden disruption in patterns over time. For the United States (1945-2005), years from 1945 and onwards were selected to allow for comparisons with results from other studies (Canudas-Romo, 2008). All available years were selected for Canada (1921-2005) and Japan (1947-2007). Moreover, focus is on mortality occurring at age ten and above because infant and child mortality present unique features that would require the use of a smoothing method suited for ill-posed data, which goes beyond the scope of this research.

Let $m_i = Y_i/E_i$ denote the death rate at age i. Under the assumption of a true constant instantaneous death rate (force of mortality) μ_i within each age and time interval, the number of deaths Y_i is a realization of a Poisson distribution with mean $E_i \cdot \mu_i$, that is

$$Y_i \sim \text{Poisson}(E_i \cdot \mu_i). \tag{1}$$

The Poisson regression model used in this paper is based on this assumption. Since our response variable $\boldsymbol{y}$ is non-normally distributed, we introduce a linear predictor $\boldsymbol{\eta}$ and the logarithm as the canonical link function, such that

$$\boldsymbol{\eta} = \ln(\mathrm{E}[\boldsymbol{y}]). \tag{2}$$

The Poisson regression model assumes $\boldsymbol{\eta}$ can be modelled by a linear combination of unknown parameters. Following the work of Eilers and Marx (1996) and Camarda (2009), we use a flexible nonparametric approach based on regression splines, specifically B-splines with penalties known as P-splines, to estimate those parameters. The method of P-splines is described in more detail in Appendix A.

From relations (1), and (2), we then have

$$\boldsymbol{\eta} = \ln(\mathrm{E}[\mathbf{y}]) = \ln(\boldsymbol{e} \cdot \boldsymbol{\mu}) = \ln(\boldsymbol{e}) + \ln(\boldsymbol{\mu}) = \ln(\boldsymbol{e}) + \boldsymbol{B}\boldsymbol{a},$$

where $\boldsymbol{y}$, $\boldsymbol{e}$, $\boldsymbol{\mu}$ are respectively observed deaths, exposures to risk, and force of mortality vectors for a given region and period. The term $\ln(\boldsymbol{e})$ is commonly referred to as the *offset* in a Poisson regression setting. Furthermore, $\boldsymbol{B}$ is the B-spline basis matrix and $\boldsymbol{a}$ is the vector of respective regression parameters to estimate. Using the method of P-splines adapted for Poisson counts to estimate $\boldsymbol{a}$, we obtain

$$\hat{\boldsymbol{\eta}} = \ln(\boldsymbol{e}) + \boldsymbol{B}\hat{\boldsymbol{a}}, \tag{3}$$

and a smoothed trend for the force of mortality is readily obtained as

$$\hat{\mu}(x) = \exp\left(\boldsymbol{B(x)}\hat{\boldsymbol{a}}\right).$$

The corresponding smoothed survival function expressed as

$$\hat{S}(x) = \exp\left(-\int_0^x \hat{\mu}(t)dt\right)$$

can then be calculated using standard numerical integration techniques. The smoothed probability density function describing the age-at-death distribution is given by

$$\hat{f}(x) = \hat{\mu}(x)\hat{S}(x).$$

The estimated modal age at death corresponding to $\hat{M} = \max_x \hat{f}(x)$, and estimated quantiles $\hat{Q}_{15}, \hat{Q}_{20}$ calculated on both sides of this estimated mode serve as indicators to monitor changes in the distribution of ages at death over time. A decline in the distance between the estimated mode and a given quantile over time indicates that deaths are being compressed along age.

Note that as pointed out by Wilmoth (1997), the three curves $\mu(x)$, $S(x)$, and $f(x)$ are related in such way that changes in one of them will necessarily be reflected in the other two. In this paper, we mainly focus on the smoothed density function describing the age-at-death distribution, but analysis based on $\hat{\mu}(x)$ or $\hat{S}(x)$ would yield consistent conclusions.

Results

Figure 1 shows fitted death counts in Japan for each sex according to model (3) for a sample of years between 1947 and 2007. Actual (observed) deaths also appear onto this figure and we can see that the Poisson regression model fits very well.

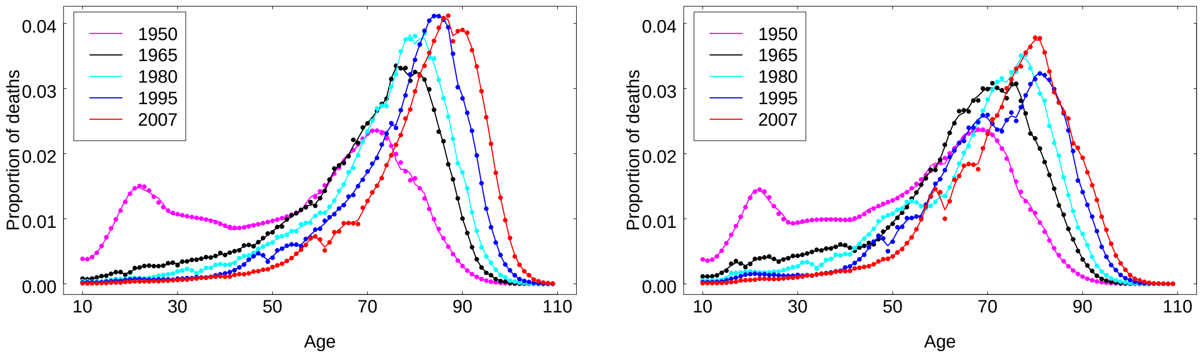

Figure 1: Actual and fitted death counts in Japan for women (left) and men (right). *Source*: HMD

The smoothed underlying forces of mortality by sex, free of exposure to risk, are shown in figure 2 for the same sample of years. The picture is not surprising: even though infant mortality is omitted, the rapid overall mortality decrease and transformation of the mortality profile by age of the second half of the twentieth century in Japan is well revealed for both sexes.

The corresponding smoothed survival functions $\hat{S}(x)$ and density functions $\hat{f}(x)$ by sex are provided in figures 3 and 4 respectively. The former clearly exihibit how the survival curves became more and more rectangular during the second half of the twentieth century. This is reflected in the latter by the compression of mortality regime during which deaths have been compressed into a narrower age interval. The density curves of figure 4 are in fact comparable to the life table deaths (see figure 8 in Appendix B), but have the advantage of being known numerically. This allows us to obtain accurate estimations of the late modal age at death and quantiles from both sides of it. The smoothed densities are also free from any modeling structure at advanced ages that would otherwise impose a strict behaviour of deaths.

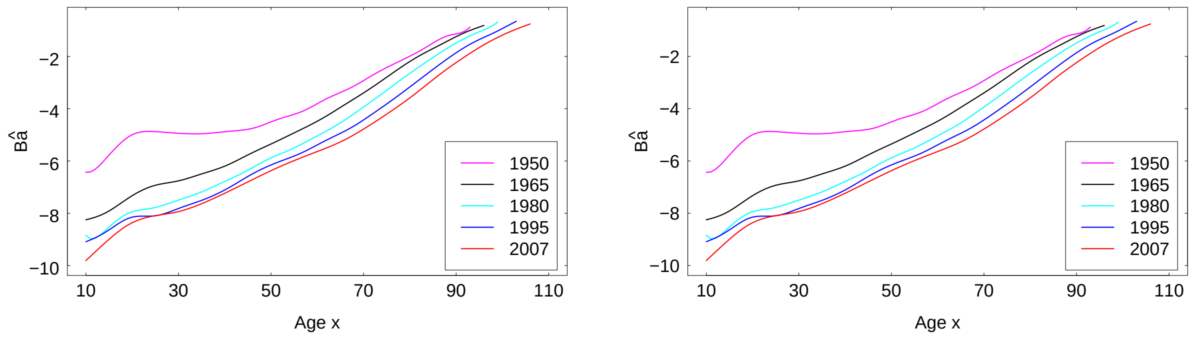

Figure 2: Smoothed force of mortality in Japan for women (left) and men (right). *Source*: HMD

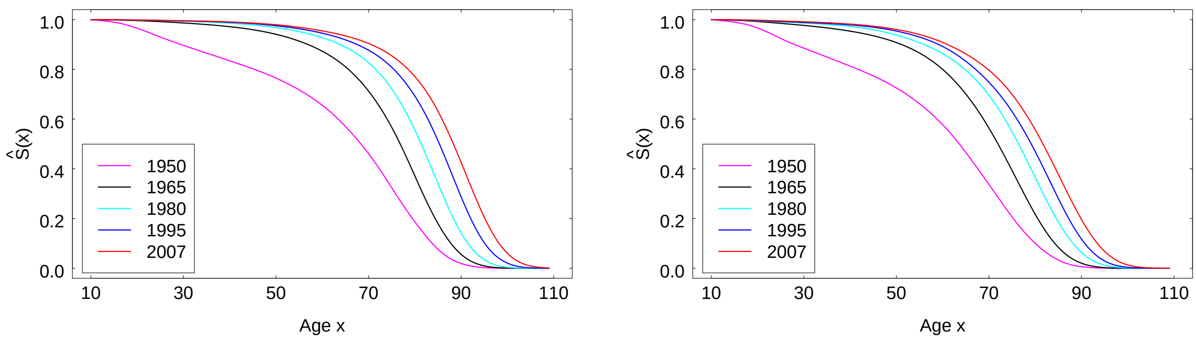

Figure 3: Smoothed survival functions in Japan for women (left) and men (right). *Source*: HMD

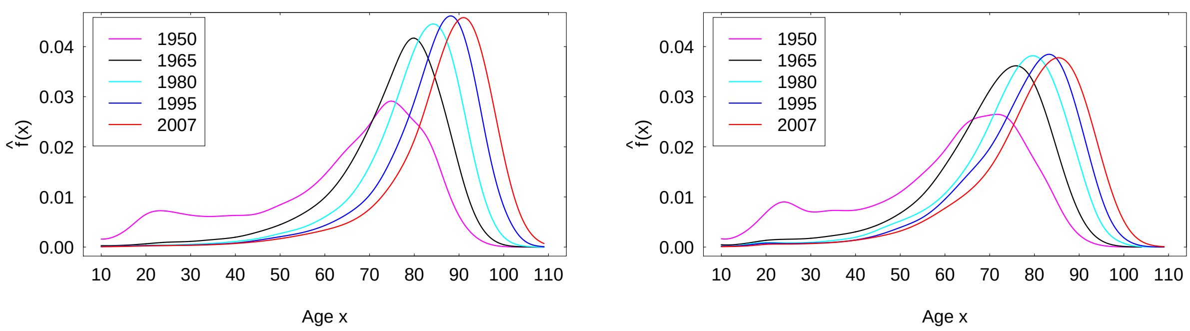

Figure 4: Smoothed density functions in Japan for women (left) and men (right). *Source*: HMD

Modal age

A simple visual inspection of 4 lets us anticipate a systematic greater modal age at death ($\hat{M}$) among women compared to men in Japan during the 1947-2007 period. However, detailed time trends are quite difficult to assess, and comparisons with other countries quickly become out of hand with such inspection. Figure 5 allows for a much better appreciation. Among women, Japan shows an upward trend which has been unquestionably linear between 1960 and 2004. The average growth rate of more than 3 months per year has contributed to bring $\hat{M}$ above 90 years in 2000. Japanese women were showing the lowest $\hat{M}$ values in the 1950s and 1960s, but their quick and steady mortality improvement brought them well above the others. However, after four decades of sustained rapid increase, results for the most recent years reveal an unexpected levelling off and their advantage over other low mortality countries is thinning.

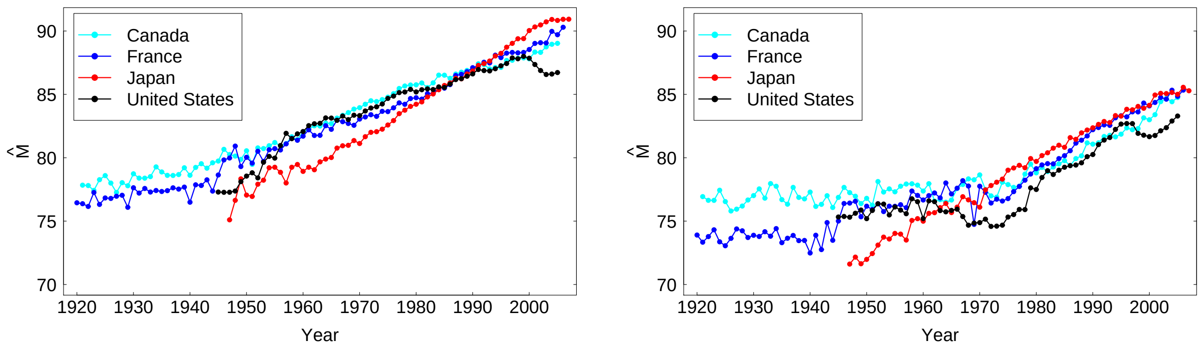

Figure 5: Estimated modal age at death in four low mortality countries for women (left) and men (right). *Source*: HMD

Indeed, although slightly less steady than for Japan, increasing linear trends for France and Canada have occurred throughout the years under study and the latest results show no sign of breathlessness. Canadian women have had a longlasting advantage over the French, but since the mid-1980s, the opposite is happening. As for the case of U.S. women, it is somehow unique: after having experienced an upward trend comparable to the one of Canadian women between the 1950s and 1990s, $\hat{M}$ then suddently decreased between 1999 and 2003. Consequently, based on the 2005 results, U.S. women substantially disadvantaged compared to the other three countries in terms of $\hat{M}$.

Based on the right panel of figure 5, similar comments to those made for Japanese women apply to Japanese men, although the slowdown in $\hat{M}$ for recent years is less pronounced. French and Japanese men have been following each other very closely since the beginning of the 1990s. Canadian men also seem to have recently joined them, even though increases in $\hat{M}$ took long a long time to start. Indeed, from the 1920s to the 1960s, little change was recorded for Canadian men, most probably because improvements in mortality at ages above $\hat{M}$ required to its increase were limited during those years (Canudas-Romo and Wilmoth, 2007). Finally, the upward trend for U.S. men has been interrupted twice: $\hat{M}$ actually decreased between 1962 and 1974, and between 1997 and 2000. Accordingly, just as is was the case with U.S. women, U.S. men fall behind the other three countries in 2005.

Quantiles above the modal age

Figure 6 displays sex-specific trends for the distance between $\hat{M}$, and the quantiles $\hat{Q}_{15}$ and $\hat{Q}_{20}$ above it. The time trend in both $\hat{Q}_{15} - \hat{M}$ and $\hat{Q}_{20} - \hat{M}$ can be used to monitor changes in the dispersion of deaths over age. Specifically, a decline in the distance between the estimated modal age and either $\hat{Q}_{15}$ or $\hat{Q}_{20}$ over time indicates that compression of mortality is occuring above the modal age and vice versa. The distance measure based on $\hat{Q}_{15}$ as opposed to $\hat{Q}_{20}$ monitors dispersion changes occurring in a smaller neighbourhood of $\hat{M}$. Other quantiles can also be selected depending on the purpose of the study.

The top-left panel of figure 6 reveals that compression of mortality above $\hat{M}$ among Japanese women was the most rapid of all four countries under study. Their strongest compression episode occurred between 1947 and 1960, and afterwards, the pace slowed down substantially. In fact, since the mid-1980s, $\hat{Q}_{20} - \hat{M}$ remained more or less the same, indicating that compression may have come to an end. French and Canadian women have experienced their strongest compression between the 1920s and the 1950s, but since the end of the 1990s, their trends in $\hat{Q}_{20} - \hat{M}$ have also stopped decreasing. For U.S. women, the process of mortality compression has been very slow since the mid-1950s and the overall declining trend in $\hat{Q}_{20} - \hat{M}$ includes long pauses. In 2005, the levels of dispersion above the modal age among women is greater in the U.S. than in Canada, but lower in France and Japan. Such ranking has in fact been holding for about four decades.

From the men's perspective, the trends in $\hat{Q}_{20} - \hat{M}$ also show that Japan experienced the fastest compression of mortality process among these countries. Furthermore, since the beginning of the 1980s, little change has been recorded for Japanese men in $\hat{Q}_{20} - \hat{M}$, suggesting that compression of mortality could be over. Canadian men have recently been

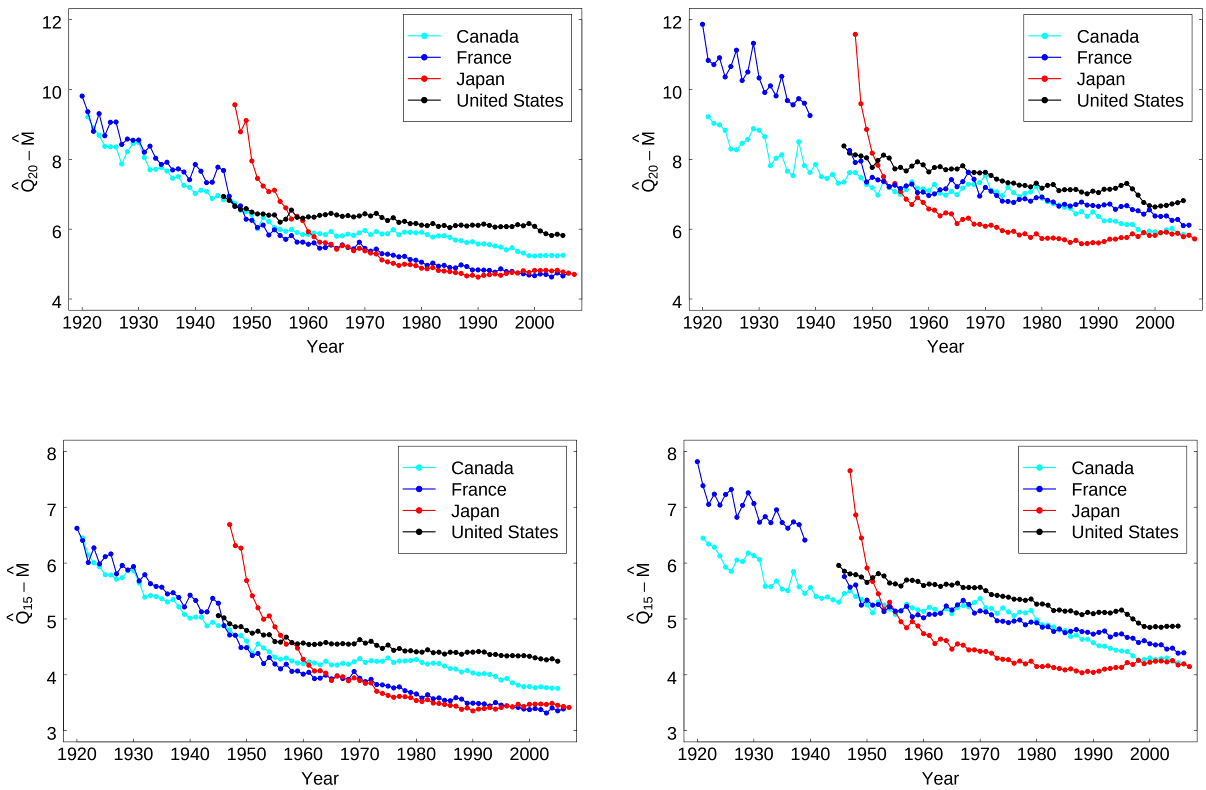

Figure 6: Distance between the estimated modal age and quantiles above it in four low mortality countries for women (left) and men (right). The years of the Second World War 1940-1945 have been excluded for French men. *Source*: HMD

following the Japanese path closely, mainly because of the steep declining trend between 1980 and 2000 (this decline was almost comparable to the pre-1960s decline). French men seem to be converging towards the Japanese, but their recent declining trend in $\hat{Q}_{20} - \hat{M}$ might as well continue. Among U.S. men, compression of mortality occurred rather smoothly between the 1950s and the beginning of the 1990s. The trend in $\hat{Q}_{20} - \hat{M}$ afterwards is incertain: convergence toward the other countries is doubtful and there are not guarantee that compression of mortality will soon come to an end.

The bottom panel of figure 6 is used to investigate whether these findings on dispersion above the modal age still hold when a smaller neighbourhood above the modal age is considered. Similar comments apply indeed, but we can also notice that the discrepancy in the level of dispersion in 2005 between U.S. women and those of other countries is even greater in the area closer to the modal age.

Discussion

Changes in the age-at-death distribution during the twentieth century and the first years of the present century were usually monitored using measures that are tied to parametric modelling methods. In this paper, we introduced a nonparametric method that has the potential to improve our ability to follow and assess changes in the central tendency and dispersion of deaths over time. As opposed to parametric method, ours does not rely on any rigid theoretical assumptions or modeling structure. Such flexibility therefore allows us to obtain a finer expression of the underlying mortality trend as described by the actual death and exposure to risk data. For developed countries with reliable data and reasonable population size, mortality changes over age show regular patterns and the use of the P-splines method represents a natural choice; it aims at obtaining a satisfactory fit to the actual data while keeping a smooth curve. Furthermore, we have showed that this nonparametric approach yields smoothed density functions that are comparable to the life table deaths while offering greater precision. Indeed, the modal age at death and quantiles from this modal age can easily be estimated accurately and used to monitor changes in the age-at-death distribution.

The countries of Canada, France, Japan and the United States were selected as a subset of low mortality countries to illustrate the new method. The key findings can be summarized as follow: in terms of modal age at death, for the first time in about four decades, the steep upward trend among Japanese has unexpectedly slowed down. In Canada and France, the steady upward trend continues, but in the U.S., substantial decreases have recently been recorded. In terms of dispersion of deaths above the mode, Japanese women are showing convincing evidence that they have come thru their rapid and intensive phase of compression of mortality. French and Canadian women seem to be going in that direction as well. For U.S. women, the pace of compression is low but may keep on going. Among men, Japan is the only country with strong evidence favoring compression of mortality slowdown. This set of results reflects that important differences across populations at comparable levels of mortality do occur.

In recent studies, authors have argued that some low mortality countries might be currently moving from a phase of compression of mortality to a new regime corresponding to the shifting mortality described by Bongaarts (2005) and Kannisto (1996). In this regime, compression has stopped and the whole distribution of adult life durations slides to higher ages over time. According to Cheung and Robine (2007) and Cheung et al. (2008), France and Italy, although not as advanced as Japan in this process, seem to be following its path.

We have found somehow consistent results. According to our study, Japanese men and women have repeatedly shown evidence that they may have come thru their compression of mortality phase. The shifting mortality regime does seem to provide a more adequate depiction of their recent mortality changes, as long as we are comfortable with interpreting the recent slowdown in the modal age as a deviation resulting from the transition period to the new regime (Canudas-Romo, 2008). The shifting mortality regime also seems to give a realistic desription of the recent situation in France and Canada among women. However, in the light of our findings, the phase of compression of mortality still defines best the situation of U.S. women and men, French men, and most probably Canadian men.

Appendix A

B-splines consist of polynomial pieces that are joined at certain abscissa values called *knots*. The degree of the polynomial is set by the user, while the number of knots and their position can be chosen according to automatic optimization schemes developed by Friedman and Silverman (1989) and Kooperberg and Stone (1991, 1992). Formulas by de de Boor (1977, 1978), Cox (1981) or Dierckx (1993) can then be used to compute the B-splines recursively.

A set of B-splines is called a *B-spline basis* and is well-suited for smoothing observed data points (x_i, z_i), $i = 1, \dots, n$. Figure 7 shows an example of a B-spline basis, which contains a set of 8 equally-spaced B-splines of degree 3, namely cubic B-splines. The basis matrix $\boldsymbol{B}$ associated with this particular B-spline basis is defined as

$$\boldsymbol{B} = \begin{bmatrix} B_1(x_1) & B_2(x_1) & \dots & B_8(x_1) \\ B_1(x_2) & B_2(x_2) & \dots & B_8(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ B_1(x_n) & B_2(x_n) & \dots & B_8(x_n) \end{bmatrix},$$

where $B_j(x_i)$, $j = 1, \dots, 8$ denotes the value at x_i of the jth cubic B-spline. A fitted curve $\hat{z}$ to observed data points (x_i, z_i) is then expressed as $\hat{z}(x_i) = \sum_{j=1}^{8} \hat{a}_j B_j(x_i)$, where $\hat{a}_j$ is the estimated regression coefficient of $B_j(x_i)$. More generally, we have $\hat{\boldsymbol{z}} = \boldsymbol{B}\hat{\boldsymbol{a}}$, meaning that we are still in the framework of classical regression.

B-splines are certainly attractive for nonparametric modeling, but the task of choosing the optimal number and positions of the knots remains a very complex one. The use of equidistant knots may be seen as a good option, but it often leads to limited control over smoothness

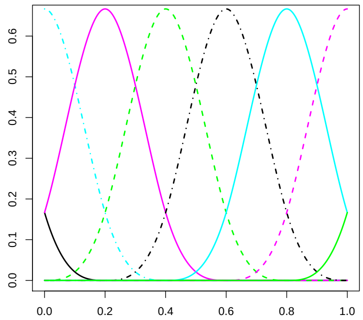

Figure 7: B-spline basis containing 8 cubic B-splines with equally-spaced knots

and fit. Inspired by the work of O'Sullivan (1988), Eilers and Marx (1996) developed the method of P-splines, which combines B-splines and difference penalties on the estimated coefficients of adjacent B-splines. The idea behind this approach is to use a relatively generous amount of equally-spaced knots, and to apply a penalty on the regression coefficients to ensure a smooth variation and avoid over-fitting. Specifically, we used one knot for every five data point, cubic B-splines, and a quadratic penalty term in the likelihood function. A smoothing parameter is included in the penalty term and it drives the trade-off between parsimony and accuracy of the model. As suggested by Currie et al. (2004), this smoothing parameter is chosen according to the Bayesian Information Criterion.

Appendix B

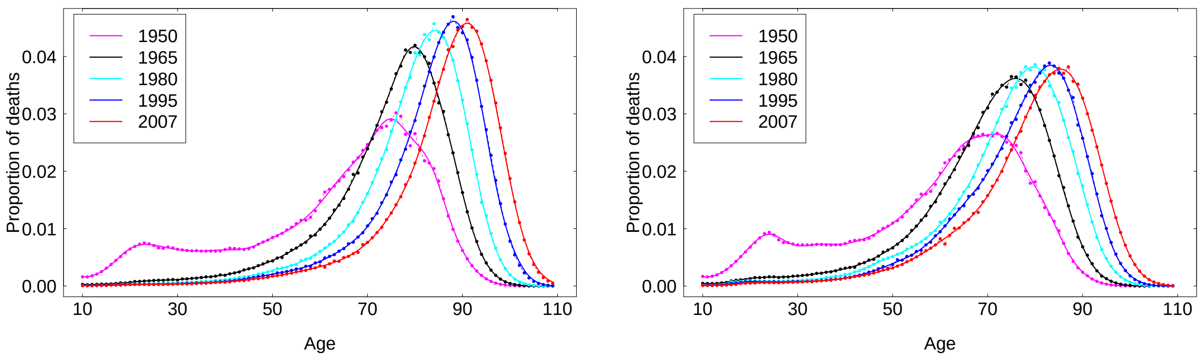

Figure 8: Smoothed density functions (solid lines) and life table deaths (dashed lines) in Japan for women (left) and men (right). *Source*: HMD

Acknowledgements

The authors would like to thank the Social Sciences and Humanities Research Council of Canada for its financial support. Part of this research was conducted while the first author visited the Statistical Demography Laboratory at the Max Planck Institute for Demographic Research. The authors are grateful to Jutta Gampe for her helpful suggestions and support, and to Carlo G. Camarda for sharing his R routines performing *P*-splines smoothing in a Poisson regression setting.

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